Programming Language and Compiler Support for Uncertainties

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Writing accurate numerical software is hard because

- math is hard
- there are many unavoidable uncertainties
  - measurement errors
  - finite precision arithmetic
  - limited resources/truncation errors

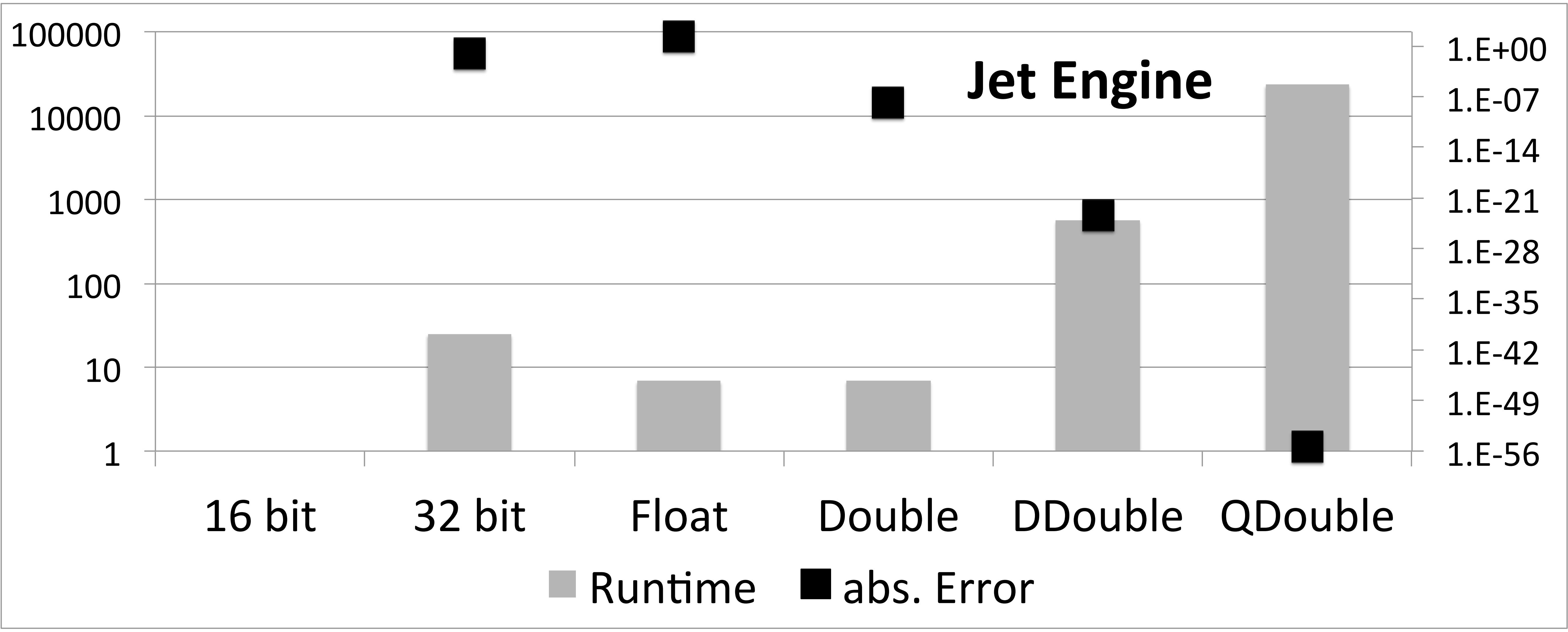
Our goal: automate reasoning about uncertainties

```
def jetEngine(x1: ???, x2: ???): ??? = {

  val t = (3*x1*x1 + 2*x2 - x1)
  val t2 = x1*x1 + 1
  x1 + ((2*x1*(t/t2)*(t/t2 - 3) + x1*x1*(4*(t/t2)-6))*t2 +
    3*x1*x1*(t/t2) + x1*x1*x1 + x1 +
    3*((3*x1*x1 + 2*x2 -x1)/t2))

}
```

```
def jetEngine(x1: Real, x2: Real): Real = {
  require(-5 <= x1 && x1 <= 5 && -20 <= x2 && x2 <= 5)

  val t = (3*x1*x1 + 2*x2 - x1)
  val t2 = x1*x1 + 1
  x1 + ((2*x1*(t/t2)*(t/t2 - 3) + x1*x1*(4*(t/t2)-6))*t2 +
    3*x1*x1*(t/t2) + x1*x1*x1 + x1 +
    3*((3*x1*x1 + 2*x2 -x1)/t2))

} ensuring(res => res <= 5200 && res +/- 1e-7)
```

```scala
def jetEngine(x1: Double, x2: Double): Double = {
  require(-5 <= x1 && x1 <= 5 && -20 <= x2 && x2 <= 5)

  val t = (3*x1*x1 + 2*x2 - x1)
  val t2 = x1*x1 + 1
  x1 + ((2*x1*(t/t2)*(t/t2 - 3) + x1*x1*(4*(t/t2)-6))*t2 +
    3*x1*x1*(t/t2) + x1*x1*x1 + x1 +
    3*((3*x1*x1 + 2*x2 -x1)/t2))

} ensuring(res => -1997.03679344963210 <= res &&
                  res <= 5109.33737094863900 &&
                  res +/- 1.6170398214864217e-08)
```

Precision vs Efficiency



Why Program in Reals?

- separation of concerns

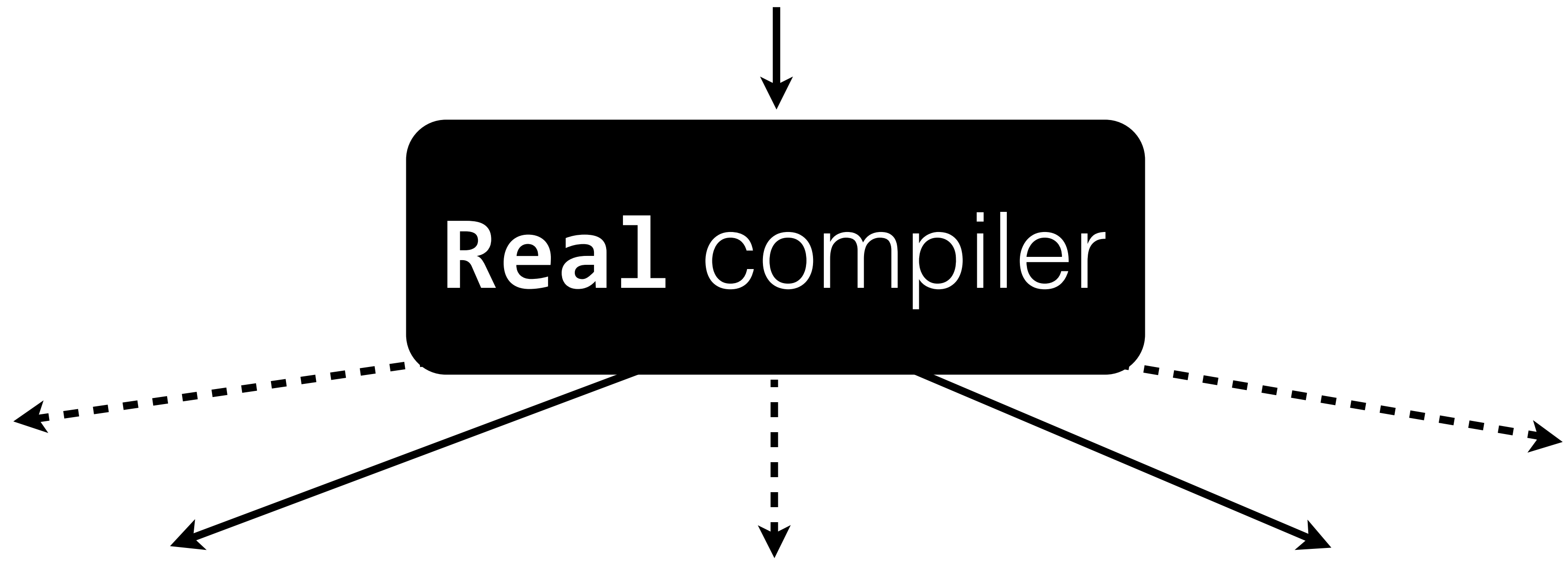
  mathematical vs implementation

- verification over reals
- compiler optimisations
- specification of ideal behaviour

  quantifying deviations

real-valued program & specification

```
def jetEngine(x1: Real, x2: Real): Real = {
  require(-5 <= x1 && x1 <= 5 && -20 <= x2 && x2 <= 5)
                          . . .
```



fixed-point program

floating-point program

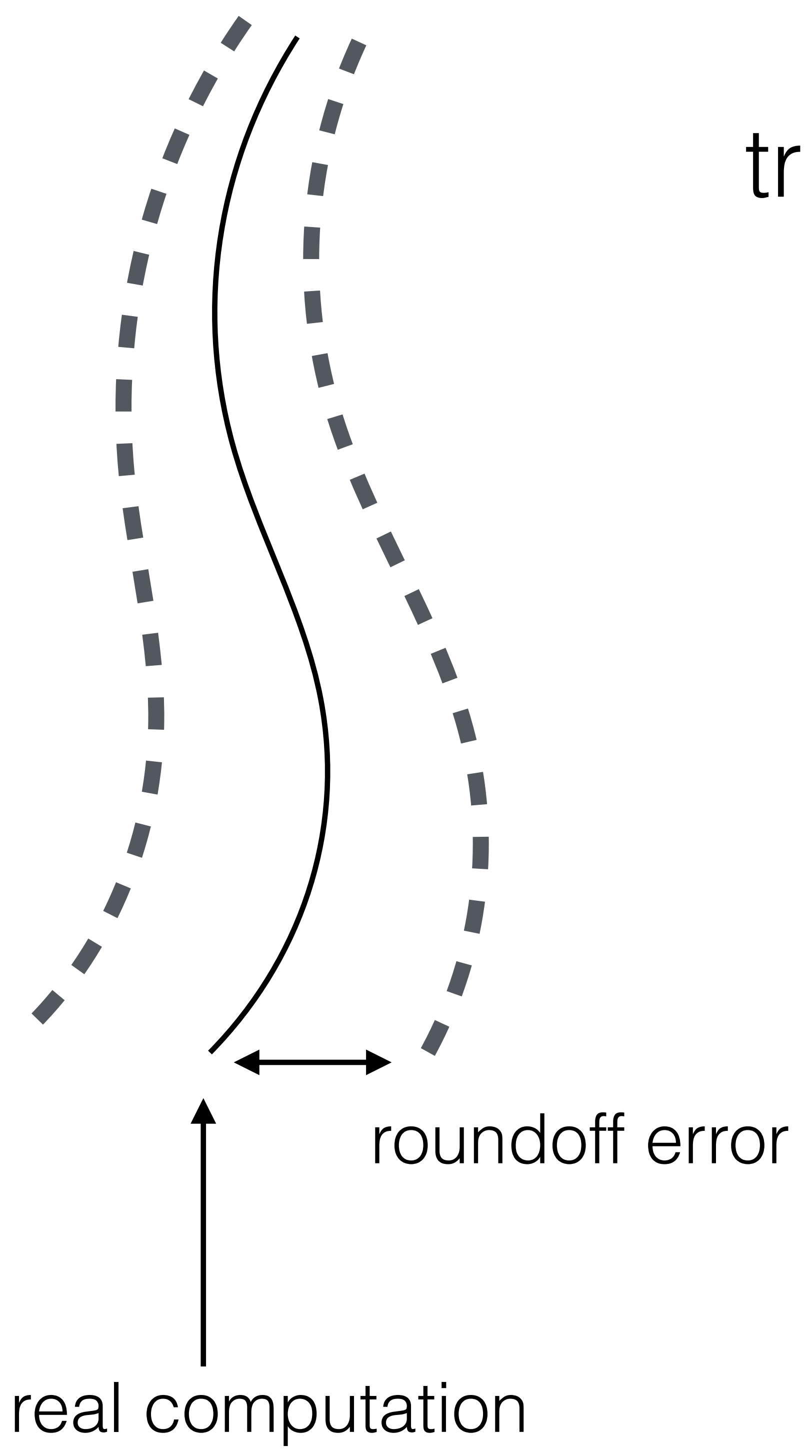
```
def jetEngine(x1: Long, x2: Long): Long = {
  val _tmp1 = ((1610612736l * x1) >> 30)
  val _tmp2 = ((_tmp1 * x1) >> 31)

                    . . .
```

```
def jetEngine(x1: Double,
              x2: Double): Double = {
  val t = (3*x1*x1 + 2*x2 - x1)

                    . . .
```

&

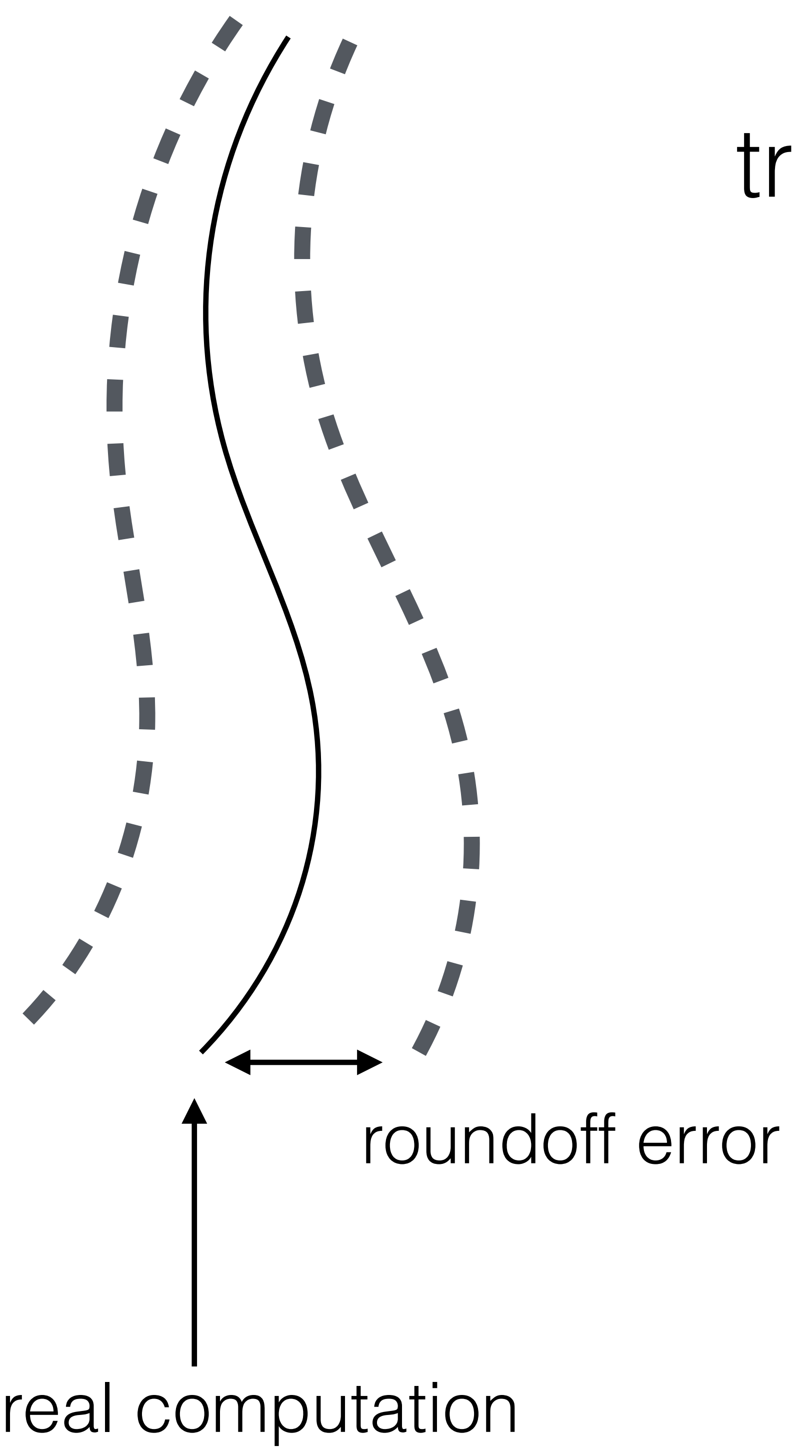
refined postcondition

Roundoff Errors



track real computation +/- uncertainties

Roundoff Errors



track real computation +/- uncertainties

affine arithmetic to maintain linear correlations

$$x = x_0 + \sum_{i=1}^{n} x_i \epsilon_i, \qquad \epsilon_i \in [-1, 1]$$

roundoff error

real computation

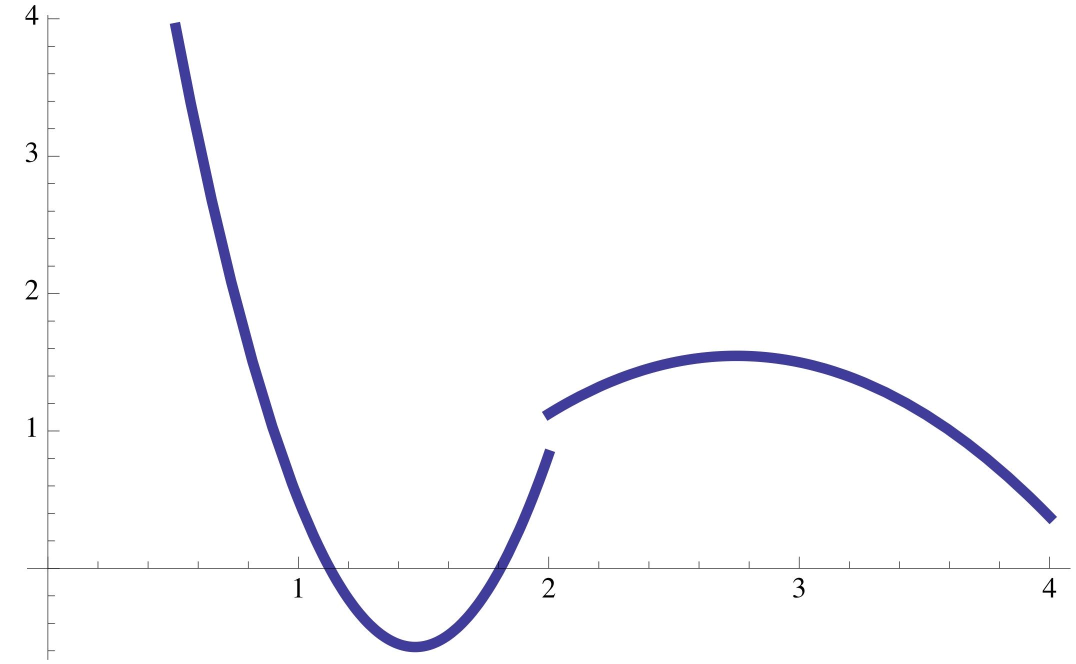
Nonlinear Arithmetic

```
def jetEngine(x1: Real, x2: Real): Real = {
  require(-5 <= x1 && x1 <= 5 && -20 <= x2 && x2 <= 5)
  val t = (3*x1*x1 + 2*x2 - x1)
  val t2 = x1*x1 + 1
  x1 + ((2*x1*(t/t2)*(t/t2 - 3) + x1*x1*(4*(t/t2)-6))*t2 +
    3*x1*x1*(t/t2) + x1*x1*x1 + x1 +
    3*((3*x1*x1 + 2*x2 -x1)/t2))
}
```

- range computation with SMT for every computation step
  - initial range with interval arithmetic
  - refine bounds with binary search with Z3
- includes any correlations between variables
- gain precision even if solver fails at some step

Conditionals

```
def approx(x: Real): Real = {
  require(0 <= x && x <= 4 && x +/- 1e-8)

  if (x <= 2) {
    10.0268 - 14.4554*x + 4.92857*x*x
  } else {
    -4.125 + 4.125*x -0.75*x*x
  }
}
```



- discontinuity error: max | then[$\mathbb{R}$] - else[$\mathbb{F}$] |
- evaluated over ``critical inputs`` ~ 2 ± 1e-8

Iterative Algorithms

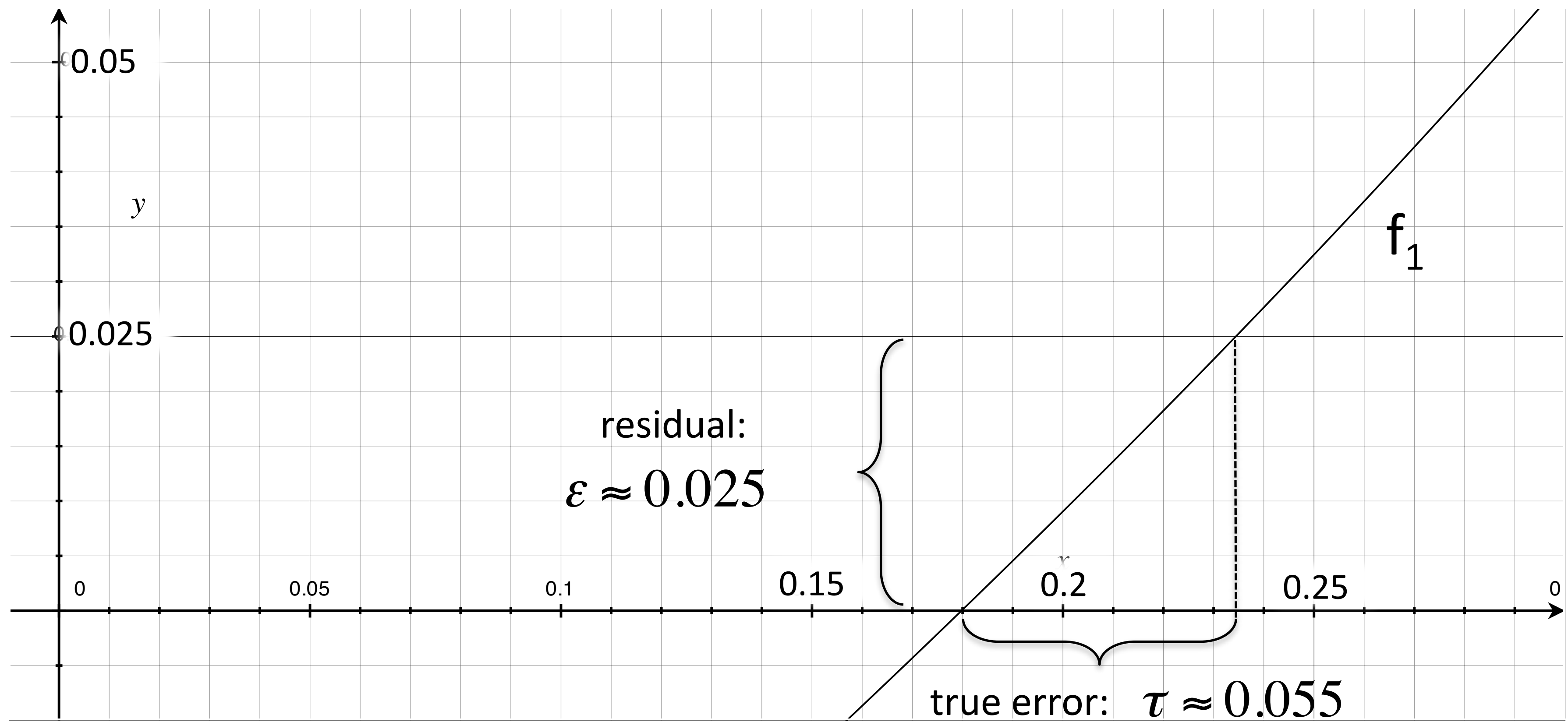
Iterative Algorithms

$\tan x_1 - k*(2*\sin x_1 + \sin x_2) = 0$

$\tan x_2 - 2*k*(\sin x_1 + \sin x_2) = 0$

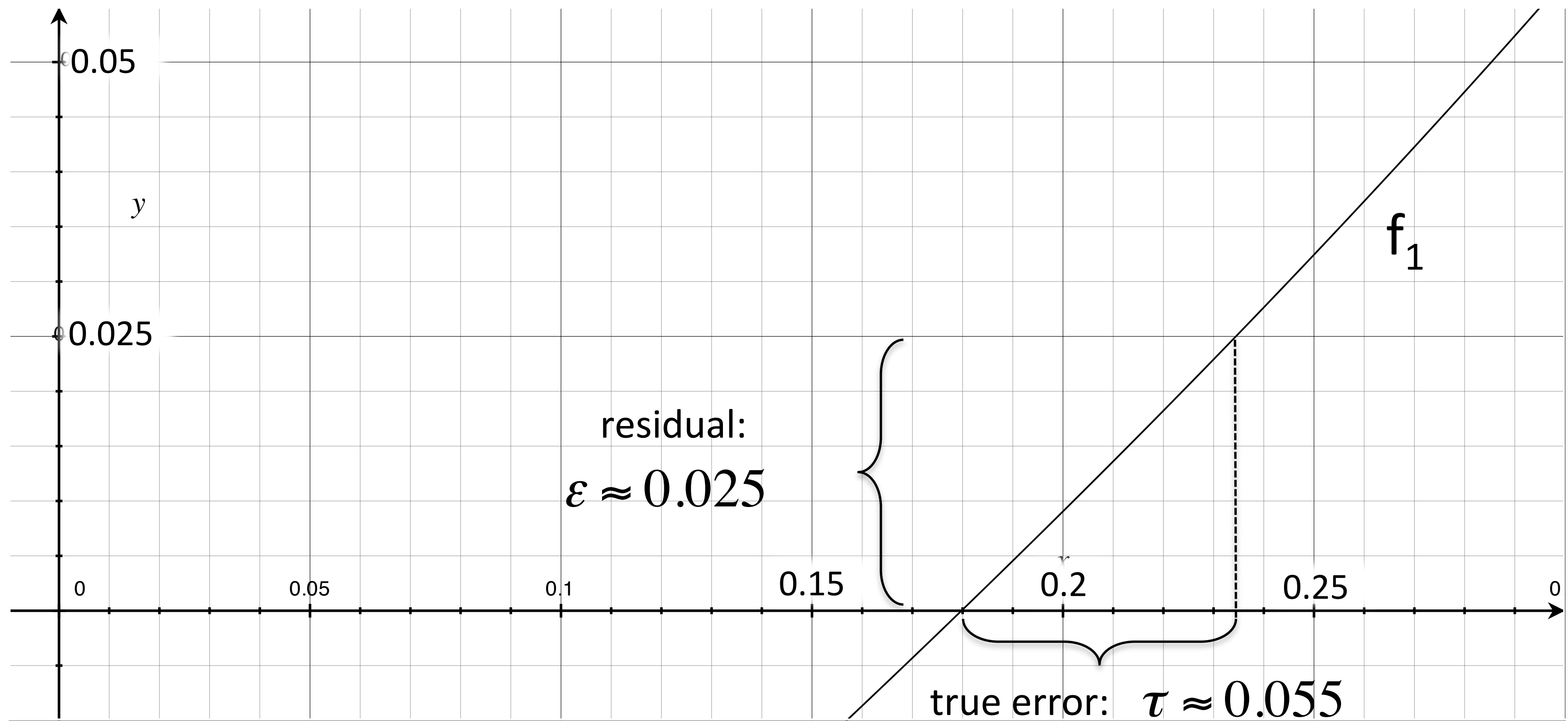
Iterative Algorithms

$\tan x_1 - k^*(2^*\sin x_1 + \sin x_2) = 0$
$\tan x_2 - 2^*k^*(\sin x_1 + \sin x_2) = 0$



Iterative Algorithms

tan x_1 – k*(2*sin x_1 + sin x_2) = 0
tan x_2 – 2*k*(sin x_1 + sin x_2) = 0



- runtime library to certify (black-box) computed results
- relies on theorems from validated computation of roots

Conclusion

Programmers

should not have to worry about low-level details of implementing real computation

only need to be aware of magnitude of errors

Reasoning can be automated to handle straight-line code, conditionals and self-stabilising iterative computations

benchmarks: physics, chemistry, biology and control systems